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Credit-based Pricing for Multi-user Class Transportation Facilities

September 2012

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DISCLAIMER

On a Credit-Based Congestion Pricing Scheme for Transportation Networks

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Abstract

This paper proposes an innovative arc-based credit (ABC) congestion pricing scheme to improve the system performance in a transportation network. By associating each arc with a positive or negative credit rate, the strategy can accomplish multiple planning goals, such as efficiency, fairness, and public acceptance simultaneously. We first demonstrate that on a one-origin or one-destination network, a *pareto-improving*, *system-optimal* and *revenue-neutral* credit scheme always exists and can be obtained by solving a set of linear equations. Recognizing that such a credit scheme may not exist in a multi-origin network, we then define the maximum-revenue problem with pareto-improving constrains (MRPI): find the maximum possible revenue collected by the credit scheme with optimal arc flows and non-increasing origin-destination (OD) travel costs. We discover that the dual of MRPI is equivalent to a typical *Transportation Problem* which, therefore, provides a simple way to calculate the revenue by just examining the dual problem. At the end of the paper, a numerical example with a small synthetic network is provided for the comparison of the credit scheme with other existing toll schemes in terms of OD travel disutilities.

Keywords: Credit-based, Self-sustainable, Pareto-improving, Revenue-neutral, No-toll equilibrium, System-optimal

1. Introduction

Fuel tax has been the primary funding source for the construction, operation and maintenance of the highway system in US since the highway trust fund was established in the 1950s. As more and more travelers drive more fuel efficient and/or alternative fuel vehicles, and a reduction of vehicle-miles traveled (VMT) under various policies to reduce the emission of greenhouse gases, the solvency of the highway trust fund in the not so distant future is called into question. Foreseeing this uncertain future of the fuel-tax based transportation financing paradigm, alternative roadway revenue strategies have recently being explored. The VMT-based charging scheme piloted in Oregon and the congestion pricing scheme implemented on

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Interstate 5 in San Diego, California are two examples. In particular, the Oregon pilot study demonstrated that a charging-by-usage revenue scheme is feasible with current technology, and that this is a viable alternative to the fuel-tax based scheme, and offers additional advantages in managing congestion. Within this context, we propose in this paper an innovative credit-based charging scheme that aims at not only improving the operating efficiency of the transportation system by reallocating traffic flow, but also reducing travelers' financial burden.

The concept and study of road pricing has a long history that dates back to the 1920s [16, 6, 12, 3]. The economic theory behind it is the use of a price mechanism to reallocate a scarce resource, road capacity, so as to reduce the efficiency loss due to travelers' selfish-routing behavior [14, 5, 13, 2, 15]. Among existing pricing schemes, marginal-cost pricing is known to achieve a socially optimal (SO) flow pattern even though travelers still minimize their own travel costs [18, 17, 1, 16]. The marginal toll is easy to calculate, although its realization may not be straightforward. This pricing scheme, however, has two obvious flaws: first, in most cases it increases the travel disutility of road users unless the tolls collected were redistributed back to the road users in some way; second, it results in two kinds of unfairness: 1) the anonymous pricing scheme usually incurs undesirable benefit distribution over the population because of road users' differences in their value-of-time, and 2) people traveling from different OD pairs may experience substantially different changes of travel disutility after the toll is imposed, depending on the levels and locations of tolls.

Motivated by reducing the total cost incurred by road users, Dial [8, 7] proposed the minimal-revenue congestion toll and provided a fast solution algorithm for obtaining the toll in a multi-origin network with fixed demand. Compared with the first-best, marginal cost toll, it lowers the total travel cost of the road users but some of the travelers may still experience an increase in travel cost. And for travelers from different OD pairs, their travel disutility may still vary significantly. Another direction to lower the travel cost is to directly return the revenue collected by the marginal-cost toll to the travelers by a monthly "credit allowance" [10, 9]. The allowance is uniformly redistributed to the drivers regardless of their travel distances. Such a redistribution scheme would pit long-distance commuters against short-distance commuters because the former subsidizes the latter, which causes another form of inequity other than price discrimination.

To address the fairness issue, "Pareto-improving" pricing schemes were introduced, which not only improve the performance of the network (not necessarily to the level of the first-best solution), but also reduce the travel disutility of travelers from every OD pair [11, 19]. Unfortunately, under general network settings, the link-based pareto-improving congestion toll scheme does not always exist if the toll rate is restricted to be nonnegative. Moreover, since the whole problem is usually formulated as an optimization problem with equilibrium and equity constraints, it is often difficult to obtain the global optimum, if there is one, in a large network.

All previously discussed road pricing schemes aim at obtaining a desirable flow distribution over the network by charging a toll on all or a selective set of links, hence increasing the travel costs of all or some of the travelers. We can call such schemes as managing traffic with a stick. Instead of forcing travelers to shift from an overly utilized road to an underutilized road with a stick (i.e. toll), can we offer them a carrot (i.e. subsidy) when they use the underutilized road? Both schemes can change travelers' route choices but the effects on the traveler's surplus are opposite. Inspired by this thought, we propose a novel carrot-and-stick strategy, called Arc-Based Credit (ABC) system, to change the flow distribution over a network. Instead of paying money to use the toll road, under the ABC system travelers can consume or earn credits when they traverse a road segment. Such a strategy requires all or part of the arcs in the network be associated with a credit rate. The credit rate can be positive, which stands for a traditional toll, or negative, which is equivalent to a subsidy.

Compared with traditional congestion tolls, the ABC system has several advantages: 1) the scheme can face less public resistance because everyone might benefit from it; 2) it adds additional degree of freedom to achieve simultaneously different planning goals, such as equity, efficiency and self-sustainability; and 3) it functions also as a revenue redistribution scheme. It is well known that allocating the revenue collected by the toll could be a controversial issue. To road users, the congestion toll is merely a variant of a lump-sum tax. In contrast, the ABC scheme is able to directly return the revenue to road users. The scheme itself can break even, which means the total credits given out to the road users could all be covered by the credits collected from the road users.

In this exploratory study we examine the properties of the ABC scheme on a network with fixed demand and identical travelers. There are several intriguing questions to be answered: does there exist an ABC scheme that can simultaneously achieve the following objectives: socially optimal, pareto-improving and revenue-neutral? If the answer is yes, does it depend on the network topology? Moreover, if it exists, can we find a fast solution algorithm to obtain it? And if not, can we relax some assumptions and obtain a second-best solution? To answer those questions, we start with discussing an alternative first-best pareto-improving credit scheme. Then the existence of a system optimal, pareto-improving and revenue-neutral ABC scheme will be proven on a single origin or single destination network. For a network with multiple origins and destinations, we develop the model and the algorithm to solve the credit scheme. Numerical examples with sample networks are also provided to demonstrate the resulting equilibrium and effectiveness of the algorithm, and to compare the credit scheme with other existing toll schemes in terms of OD travel disutility.

2. Definitions

We consider a directed network $G = (N, A)$, denoted by a set of consecutively numbered nodes, N, a set of consecutively numbered arcs, A and a set of OD pairs, W. A directed arc (i, j) has two endpoints i and j. On this network, each OD pair is connected by a set of paths through the network. The OD demands are denoted by a column vector d with entries $d_w, w \in W$.

Let $x = (x_{ij})_{ij\in A}$ and $t = (t_{ij})_{ij\in A}$ represent the arc flow and arc travel time vectors, we have $t_{ij} = t_{ij}(x_{ij})$. For the no-toll equilibrium (NTE) to be unique, we assume $t_{ij}(\cdot)$ is strictly increasing and convex. $\rho = (\rho_{ij})_{ij\in A}$ is denoted as the column vector of arc-based credit rates. When ρ_{ij} is positive, users traveling on arc (i, j) will consume ρ_{ij} amount of credit, while if ρ_{ij} is negative, users will receive $|\rho_{ij}|$ amount of credit for using arc (i, j). $c_{ij} = t_{ij}(x_{ij}) + \rho_{ij}$, represents the generalized arc travel cost (Here without loss of generality we assume that the factor to transfer time to credit, which we call the "equivalent credit value of time", is equal to 1). We also define μ as the column vector of OD travel disutilities.

Throughout the paper, the superscript "m" represents the marginal-cost (MC) based solution, "$\backsim$" over the symbol represents system-optimal (SO) solution and "$-$" over the symbol represents the solution under NTE.

3. The Alternative System-Optimal(SO) and Pareto-Improving (PI)Credit Scheme

It's well known that if the credit of every arc is charged to its marginal travel cost, the arc flow pattern is system-optimal (SO), i.e. the total system cost is minimized. The credit internalizes the external cost of road users and is always nonnegative, which takes the form

$$\rho_{ij}^{m} = x_{ij}\frac{dt_{ij}(x_{ij})}{dx_{ij}} \geq 0 \tag{1}$$

However, if we do not require the credit rate to reflect the marginal cost, there are an infinite number of credit schemes that can produce the optimal arc flows when the OD demands are fixed. Given any SO credit scheme $\tilde{\rho}$, one straightforward way to obtain an alternative credit scheme is to proportionally change all the arcs' travel costs, following the equation below

$$\rho(\alpha) = (\alpha - 1)(\tilde{t} + \tilde{\rho}) + \tilde{\rho}, \qquad 0 < \alpha \leq 1 \tag{2}$$

Under such a credit scheme, the arc travel cost becomes

$$c = \tilde{t} + \rho(\alpha) = \alpha(\tilde{t} + \tilde{\rho}) \tag{3}$$

Here we assume $0 < \alpha \leq 1$. When $\alpha = 1$, the credit pattern reduces to original system-optimal credit scheme, $\tilde{\rho}$; and when $\alpha \to 0$, all the credit rates become negative and all the generalized arc travel costs approach to 0. From eqn.(3) we observe that all the arc travel costs under the alternative credit scheme are changed by the same percentage, α. Correspondingly, all the path travel costs are also changed by α, so that the equilibrium remains unchanged. Therefore, the arc flows are still optimal under the credit scheme defined by eqn.(2). With this observation, we give Lemma 1

Lemma 1. *On a general network with fixed OD demands, we can always find a credit scheme that is pareto-improving and system-optimal. Furthermore, under such a credit scheme, every arc retains a positive generalized travel cost.*

Proof. To prove Lemma 1 we only need to construct such a credit scheme based on a general network. Suppose under some credit scheme, $\tilde{\rho}$, the system is optimal and every arc's generalized travel cost is positive (for example, the marginal-cost credit scheme). If all the OD travel disutilites under $\tilde{\rho}$ are less than the OD travel disutilites under NTE, the SO credit scheme $\tilde{\rho}$ is already pareto-improving. Otherwise, there must exist one OD pair which experiences a travel disutility increase under $\tilde{\rho}$, i.e.

$$min\{\frac{\bar{\mu}_w}{\tilde{\mu}_w}, w \in W\} < 1 \tag{4}$$

We also know that as long as a credit scheme $\rho(\alpha)$ follows eqn.(2), it is system-optimal because it will not change the arc flow pattern under $\tilde{\rho}$; To be pareto-improving, let

$$\alpha = min\{\frac{\bar{\mu}_w}{\tilde{\mu}_w}, w \in W\} \tag{5}$$

Since we reduce the credit rate on every arc proportionally until the travel disutility of the OD pair which experiences the highest percentage increase of travel disutility is equal to the NTE travel disutility, all the other OD pairs will thus experience nonincreasing travel disutility under $\rho(\alpha)$, which indicates that $\rho(\alpha)$ is pareto-improving. And because α is always greater than 0, from eqn.(3) we know that the generalized travel cost of every arc under $\rho(\alpha)$ is still positive. □

Though a SO and PI credit scheme can be easily derived in this way, their flaws are obvious: first, the spatial inequity still exists, because such a credit scheme changes all the OD travel disutility by the same percentage. Due to the OD distribution and network topology, the travel disutility changes from NTE to SO among different OD pairs may still vary significantly; second, decreasing all the arc costs by a same percentage could come with a price: considering the extreme case that the lowest OD demand experiences the highest percentage of travel disutility increase under the marginal-cost credit scheme, to be pareto-improving, we have to subsidize all the other OD pairs by giving out credits to a large number of road users who just experience very small percentage of travel disutility increase or even travel disutility decrease. In the case that the credit can finally be cashed out by travelers, the revenue collected by the credit scheme could easily go negative so that for supporting such a credit scheme, a large amount of external subsidy may be needed. Thus one may ask that can we find a credit scheme which is both *pareto-improving* and *self-sustainable* (the revenue is nonnegative)? We shall answer this question in the following sections, by first looking into the case of single-origin networks, then extending the result to multi-orgin, multi-destination networks.

4. The Case of a Single-Origin Network

4.1. Finding the SO-NTE credit scheme

We first define a credit scheme as a SO-NTE credit scheme under which *the arc flow pattern is system-optimal but the OD travel disutilities are the same with those under NTE.* As long as the NTE is not fully efficient, we know that the revenue of SO-NTE credit scheme is always positive. We also notice that according to eqn.(2) we can always reduce the credit rate on each arc in the network so as to return the revenue to travelers. Therefore, once we find a SO-NTE credit scheme, we can always find an alternative credit scheme that is pareto-improving and revenue-neutral.

Lemma 2. *There always exists one SO, PI and revenue-neutral credit scheme, $\hat{\rho}$, when the network has only one OD pair.*

Proof. We know that the system cost under marginal-cost credit scheme is always less than or equal to the system cost under NTE. That is

$$t(\tilde{x})^T\tilde{x} \leq t(\bar{x})^T\bar{x} \tag{6}$$

Because there is only one OD pair, the OD travel disutility under NTE equals

$$\bar{\mu} = \frac{1}{d}t(\bar{x})^T\bar{x} \tag{7}$$

And the OD travel disutility under the marginal-cost credit scheme becomes

$$\tilde{\mu} = \frac{1}{d}(t(\tilde{x})^T \tilde{x} + (\rho^m)^T \tilde{x}) \tag{8}$$

If $\tilde{\mu} \leq \bar{\mu}$, the marginal-cost credit scheme is just the scheme we want to find, with $\alpha = 1$. Otherwise, from eqns.(7) and (8) we have

$$(\rho^m)^T \tilde{x} > 0 \tag{9}$$

Let

$$\alpha = \frac{t(\tilde{x})^T \tilde{x}}{t(\tilde{x})^T \tilde{x} + (\rho^m)^T \tilde{x}} \tag{10}$$

Under a credit scheme defined by (2) we know that now the OD travel disutility

$$\hat{\mu} = \alpha \frac{1}{d}(t(\tilde{x})^T \tilde{x} + (\rho^m)^T \tilde{x}) = \frac{1}{d} t(\tilde{x})^T \tilde{x} < \bar{\mu} \tag{11}$$

which implies that the credit scheme is pareto-improving. And from the previous discussion, since $\hat{\rho}$ satisfies definition (2), it will not change the flow pattern under the marginal-cost credit, which means the arc flow pattern is system-optimal. Moreover, utilizing (2) and (10), the total revenue collected by $\hat{\rho}$, R, can be calculated by

$$R = \hat{\rho}^T \tilde{x} = (\alpha - 1) t(\tilde{x})^T \tilde{x} + \alpha (\rho^m)^T \tilde{x} = 0 \tag{12}$$

which implies that the credit scheme $\hat{\rho}$ is revenue-neutral. The proof is complete.

□

Actually in the one OD case, by proportionally lowering down the credit level on every arc, the system cost saved by the credit scheme is totally returned to the road users, so that every road user is able to experience a reduction of travel disutility. However, if there are multiple OD pairs, since some of the arc flows are shared by different OD pairs, it is not easy to find an anonymous arc-based credit scheme that is SO, PI and revenue-neutral. Furthermore, if the number of OD pairs are sufficiently large, such a credit scheme may not even exist. In the following, we will first discuss a special network, the single-origin or single-destination network, and then study the general situation with multiple origins and destinations.

4.2. The SO-NTE credit scheme on a single-origin network

We associate every node in the network with a node potential, π_i. π_i can be calculated by the definition that $\pi_j - \pi_i$ is equal to the length of the shortest path distance from node i to node j. We can see that under this definition, all the other nodes' potentials are relative values, depending on the node petential of the origin node, π_o, which can be arbitrarily chosen. If π_w is the node potential of the destination node w, from the definition, $\pi_w - \pi_o = \mu_w$ is the OD travel disutility of the OD pair w under equilibrium. For a given set of node potentials π, we define the reduced cost of an arc (i, j) as $c^{\pi}_{ij} = (t_{ij} + \rho_{ij}) + \pi_i - \pi_j$.

We know that if a credit scheme reproduces the SO arc flow pattern, $\tilde{x}$, the reduced costs have to be zero on the flow-bearing arcs and nonnegative on unused arcs, that is

$$\begin{cases} c_{ij}^{\pi} = 0, & if\ \tilde{x}_{ij} > 0 \\ c_{ij}^{\pi} \geq 0, & Otherwise \end{cases} \tag{13}$$

To find the SO-NTE credit scheme (if it does exist), we construct the following Maximum-Revenue problem with Pareto-Improving constraints (MRPI)

$$\max_{(\rho,\pi)} \sum_{ij} \tilde{x}_{ij}\rho_{ij} \tag{14}$$

subject to

$$\begin{aligned} \tilde{t}_{ij} + \rho_{ij} + \pi_i - \pi_j &= 0, \forall \tilde{x}_{ij} \geq 0, (i,j) \in A & (15) \\ \tilde{t}_{ij} + \rho_{ij} + \pi_i - \pi_j &\geq 0, \forall \tilde{x}_{ij} = 0, (i,j) \in A & (16) \\ \pi_w - \pi_o &\leq \bar{\mu}_w, \forall w \in W & (17) \end{aligned}$$

The objective function is maximizing the total revenue collected by the credit scheme. (15)-(16) restrict the network flow pattern to be optimal. Constraint (17) states that the credit scheme ρ has to be pareto-improving. The revenue reaches its upper-bound when constraint (17) is binding, indicating that if the solution to the problem is its upper-bound, the solution is just the SO-NTE credit scheme we are looking for. Otherwise, the SO-NTE credit scheme does not exist. To transform the greater-than-or-equal constraints to equality constraints, we define $s_{ij} \geq 0$ as a nonnegative slack variable associated with each arc $(i,j) \in A$ and $\lambda_w \geq 0$ as a nonnegative slack variable associated with each destination node $w \in W$. In addition, we define a zero-flow indicator z_{ij} to be

$$z_{ij} = \begin{cases} 1, & if\ x_{ij} = 0, \\ 0, & Otherwise. \end{cases} \tag{18}$$

By utilizing the zero-flow indicator and slack variables , the problem can be rewritten as **LP1:**

$$\max_{(\rho,\pi,s,\lambda)} \sum_{ij} \tilde{x}_{ij}\rho_{ij} \tag{19}$$

subject to

$$\begin{aligned} \tilde{t}_{ij} + \rho_{ij} + \pi_i - \pi_j - z_{ij}s_{ij} &= 0, \forall (i,j) \in A & (20) \\ \bar{\mu}_i - \pi_i - \lambda_i &= 0, \forall i \in W & (21) \\ s, \lambda &\geq 0 & (22) \\ \rho, \pi &\in \mathcal{R} & (23) \end{aligned}$$

We notice that here the credit rates and node potentials are unconstrained, which means that by solving LP1, the arc generalized travel cost could be negative if the credit rate goes negative and sufficiently low. Then naturally, one may ask is it possible that the credit scheme induces some negative cycle? To answer the question we give the lemma below

Lemma 3. *Under the ABC scheme provided by LP1, the network contains no negative cycle.*

Proof. From the definition of reduced cost c^{π}_{ij}, for any directed cycle P, we have

$$\begin{aligned}\sum_{(i,j)\in P} c^{\pi}_{ij} &= \sum_{(i,j)\in P}((t_{ij}+\rho_{ij})+\pi_i-\pi_j) \\ &= \sum_{(i,j)\in P}(t_{ij}+\rho_{ij})+\sum_{(i,j)\in P}(\pi_i-\pi_j) \\ &= \sum_{(i,j)\in P}(t_{ij}+\rho_{ij})\end{aligned} \tag{24}$$

From eqn.(21) we know that at the equilibrium, $c^{\pi}_{ij} = z_{ij}s_{ij} \geq 0, \forall (i,j) \in A$, which implies that $\sum_{(i,j)\in P} c^{\pi}_{ij} \geq 0$. From eqn.(24) we have that $\sum_{(i,j)\in P}(t_{ij}+\rho_{ij}) = \sum_{(i,j)\in P} c^{\pi}_{ij} \geq 0$, thus we conclude that as long as the ABC scheme ρ is the solution of LP1, the network does not contain a negative cycle.

□

Obviously, the upper-bound of LP1 is equal to the difference of system costs between NTE and SO.

4.3. Dual linear program

To find the dual problem of LP1, we first restate LP1 in matrix notation. Let

$I = |A| \times |A|$, the identity matrix;
$J = |W| \times |W|$, the identity matrix;
$\Delta = (\delta_{l,ij})^{l\in N}_{ij\in A}$, the network node-arc incidence matrix, where

$$\delta_{l,ij} = \begin{cases} 1, if\ i = l, \\ -1, if\ j = l, \\ 0, otherwise. \end{cases}$$

$b = \Delta\tilde{x} = (b_n)_{n\in N}$ =trip demand to each node n. We thus have

$$\begin{cases} b_i > 0, \text{if } i \text{ is an origin node,} \\ b_i < 0, \text{if } i \text{ is a destination node,} \\ b_i = 0, \text{otherwise.} \end{cases}$$

$\Lambda = (\sigma_{i,j})^{i\in N}_{j\in W}$, the diagonalization of the network node-OD incidence matrix, where

$$\sigma_{i,j} = \begin{cases} 1, \text{if } i \text{ is an origin node,} \\ -1, \text{if } i \text{ is a destination node,} \\ 0, otherwise. \end{cases}$$

From the above definitions, we immediately have

$$\Lambda d = \Delta\tilde{x} = b \tag{25}$$

$Z = (z_{a,a'})^{a\in A}_{a'\in A}$, the diagonalization of z indicators, where

$$z_{a,a'} = \begin{cases} z_a, if a = a', \\ 0, otherwise. \end{cases}$$

Then the primal problem LP1 can be rewritten as

$$\max_{(\rho,\pi,s,\lambda)} \tilde{x}^T \rho \tag{26}$$

subject to

$$\begin{pmatrix} I & \Delta^T & -Z & 0 \\ 0 & -\Lambda^T & 0 & J \end{pmatrix} \begin{pmatrix} \rho \\ \pi \\ s \\ \lambda \end{pmatrix} = \begin{pmatrix} -\tilde{t} \\ \bar{\mu} \end{pmatrix} \tag{27}$$

$$s, \lambda \geq 0 \tag{28}$$

$$\rho, \pi \in \mathcal{R} \tag{29}$$

The dual problem to eqns.(26)-(29) is

DP1:

$$\min P(y_1, y_2) = -\tilde{t}^T y_1 + \bar{\mu}^T y_2 \tag{30}$$

subject to

$$\begin{pmatrix} I & 0 \\ \Delta & -\Lambda \\ -Z & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \begin{matrix} = \tilde{x} \\ = 0 \\ \geq 0 \\ \geq 0 \end{matrix} \tag{31}$$

From the first constraint we readily obtain that $y_1 = \tilde{x}$. The fourth constraint restricts y_2 to be nonnegative. And since $y_1 = \tilde{x}$, $-Zy_1 = 0$ always holds. Thus the third constraint is redundant. The second constraint gives that $\Lambda y_2 = \Delta y_1 = \Delta \tilde{x} = b$. We know that $|\Lambda| = |y| =$ the number of Destination nodes, thus y_2 is unique. From eqn.(25), y_2 is just equal to the OD demand vector d. Substituting all the constraints, the objective function becomes

$$P(y) = \bar{\mu}^T d - \tilde{t}^T \tilde{x} \tag{32}$$

The first term in eqn.(32) is the total travel cost under user equilibrium and the second term is the total travel cost under SO. Thus the result of primal linear program LP1 is always the upper-bound of the problem,which implies that the SO-NTE credit scheme always exists, as well as the SO, PI and revenue-neutral credit scheme.

The proof is similar for the single-destination multi-origin network. The algorithm is exactly the same except that we need to set the potential of the destination node to be 0 and the potentials of other nodes as the shortest path distance from this node to the destination node. To sum up, we give the following theorem

Theorem 1. *On a single-origin or single-destination network, we can always find a credit scheme, $\hat{\rho}$, which has the following properties:*

i) Every road user experiences the same percentage improvement of the travel disutility;

ii) The arc flow pattern is socially optimal;

iii) The revenue collected by the credit is 0*, i.e. the credit scheme is break-even (i.e., revenue neutral).*

Under the SO-NTE credit scheme given by LP1 we have

$$\lambda_i = 0, \forall i \in W \tag{33}$$

Thus the SO-NTE scheme can be solved by the following group of equations

$$\begin{cases} \tilde{t}_{ij} + \rho_{ij} + \pi_i - \pi_j = 0, \forall x_{ij} > 0 \\ \pi_i = \bar{\mu}_i, \forall i \in W \\ \pi_o = 0 \end{cases} \tag{34}$$

The extreme situation happens when every node in the subgraph except the origin node is a destination node. Then the SO-NTE credit scheme is unique.

Theorem 1 gives us some good news: compared with the minimal-revenue pricing scheme proposed by Dial [7], the advantages of the credit scheme here are obvious: it not only further lowers the revenue and as a consequence lowers the travel cost of each traveler, but also mitigates OD-specific price discrimination. Furthermore, the credit scheme does not require every arc to be associated with a credit when the number of destination nodes is relatively small. Assume the number of nodes in the network is n and w of them are destination nodes, then the freedom of the problem (34) is $(n-1-w)$, which means at most $(n-1-w)$ arcs can be free arcs.

4.4. A numerical example

Here we provide a numerical example (Figure 1) of the famous network of Braess' Paradox [4].

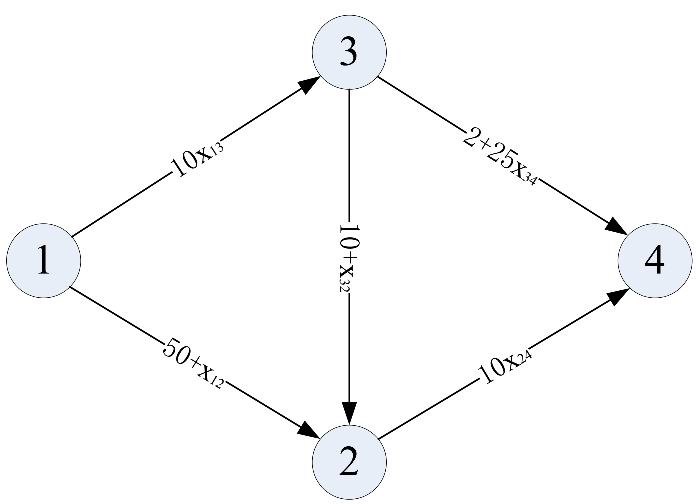

Figure 1: Network of Braess' paradox

Suppose node 1 is the origin node, nodes $2, 3, 4$ are all destination nodes. The OD demands $d_{12} = 2, d_{13} = 1.5, d_{14} = 4$. Solving SO and NTE we have the optimal arc travel time and OD travel disutility under NTE

$$\begin{pmatrix} \tilde{t}_{13} \\ \tilde{t}_{12} \\ \tilde{t}_{34} \\ \tilde{t}_{24} \\ \tilde{t}_{32} \end{pmatrix} = \begin{pmatrix} 26.96 \\ 54.80 \\ 31.89 \\ 28.04 \\ 10.00 \end{pmatrix}, \qquad \begin{pmatrix} \bar{\mu}_{14} \\ \bar{\mu}_{12} \\ \bar{\mu}_{13} \end{pmatrix} = \begin{pmatrix} 79.22 \\ 53.30 \\ 42.01 \end{pmatrix} \tag{35}$$

We also have the total system costs under NTE and SO, which are respectively 486.47 and 452.74. Setting $\alpha = 452.74/486.47 = 0.93$. We calculate the objective OD travel disutility under the pareto-improving and revenue-neutral credit scheme

$$\begin{pmatrix} \hat{\mu}_{14} \\ \hat{\mu}_{12} \\ \hat{\mu}_{13} \end{pmatrix} = \begin{pmatrix} 73.72 \\ 49.60 \\ 39.10 \end{pmatrix} \tag{36}$$

which are also the node potentials for destination nodes $2, 3, 4$ (we set the node potential of the origin node 1 to be 0). By solving a group of linear equations, we obtain the pareto-improving, system-optimal and revenue-neutral credit scheme, which are shown in the parentheses in Figure 2. The underlined number associated with each node is the node potential under the credit scheme.

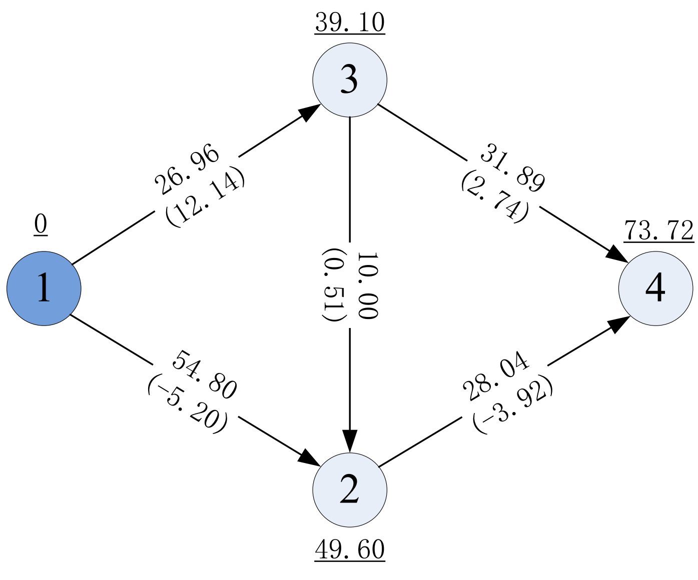

Figure 2: Resulting credit rates

5. The Case of Multi-Origin Networks

5.1. The MRPI problem

Similar with the one-origin network, for the multi-origin network we can still formulate the MRPI problem for finding a SO and PI credit scheme with the lowest possible external subsidy required. The SO and PI credit scheme is not necessarily self-sustainable now, since we have only one single set of credit rates to deal with the trips from all the origins. However, in case the maximum revenue is negative, it's still meaningful to look for a SO and PI credit scheme, with the minimum possible subsidy required, which is able to eliminate the deadweight loss of the selfish-routing behavior and make everyone better-off simultaneously. By using superscript k as origin index, we first enrich the notations as below

K	: origin-node set,
π^k	: $(\pi_n^k)_{n\in N}^{k\in K}$, node potentials for origin k,
s^k	: $(s_{ij}^k)_{ij\in A}^{k\in K}$, arc slack variables for origin k,
λ^k	: $(\lambda_w^k)_{w\in W^k}^{k\in K}$, OD slack variables for each destination nodes with origin k,
b^k	: $(b_n^k)_{n\in N}^{k\in K}$, the OD matrix,
Λ^k	: $(\sigma_{i,j}^k)_{j\in W^k}^{i\in N}$, the diagonalization of the network node-OD incidence matrix for origin k,
Z^k	: $(z_{a,a'}^k)_{a'\in A}^{a\in A}$, the diagonalization of z^k indicators,

Without loss of generality,we investigate the two-origin case, since the general pattern for n origins can then be easily revealed from the two-origin case. The primal linear program for the two-origin case can be written in matrix notation in the following form. Here we add a reasonable assumption that the generalized arc travel cost after the credit scheme cannot be negative, i.e. $\rho + \tilde{t} \geq 0$, so that people will not experience lower travel cost when they travel longer.

$$\max_{(\rho,\pi,s,\lambda)} \tilde{x}^T \rho \tag{37}$$

subject to

$$\begin{pmatrix} I & \Delta^T & 0 & -Z^1 & 0 & 0 & 0 \\ I & 0 & \Delta^T & 0 & -Z^2 & 0 & 0 \\ 0 & (-\Lambda^1)^T & 0 & 0 & 0 & J^1 & 0 \\ 0 & 0 & (-\Lambda^2)^T & 0 & 0 & 0 & J^2 \end{pmatrix} \begin{pmatrix} \rho \\ \pi^1 \\ \pi^2 \\ s^1 \\ s^2 \\ \lambda^1 \\ \lambda^2 \end{pmatrix} = \begin{pmatrix} -\tilde{t} \\ -\tilde{t} \\ \bar{\mu}^1 \\ \bar{\mu}^2 \end{pmatrix} \tag{38}$$

$$s, \lambda \geq 0 \tag{39}$$

$$\rho + \tilde{t} \geq 0 \tag{40}$$

Let $\theta = \rho + \tilde{t} \geq 0$, which is the generalized arc travel cost. Substituting θ, we have the equivalent formulation

LPn:

$$\max_{(\theta,\pi,s,\lambda)} \tilde{x}^T \theta \tag{41}$$

subject to

$$\begin{pmatrix} I & \Delta^T & 0 & -Z^1 & 0 & 0 & 0 \\ I & 0 & \Delta^T & 0 & -Z^2 & 0 & 0 \\ 0 & (-\Lambda^1)^T & 0 & 0 & 0 & J^1 & 0 \\ 0 & 0 & (-\Lambda^2)^T & 0 & 0 & 0 & J^2 \end{pmatrix} \begin{pmatrix} \theta \\ \pi^1 \\ \pi^2 \\ s^1 \\ s^2 \\ \lambda^1 \\ \lambda^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \bar{\mu}^1 \\ \bar{\mu}^2 \end{pmatrix} \tag{42}$$

$$\theta, s, \lambda \geq 0 \tag{43}$$

From Lemma 1 we know that the SO and PI credit scheme always exists and the problem has an upper-bound which is equal to the difference of system travel time costs between NTE and SO. Thus the linear program LPn and its dual are both feasible and bounded and have the same value. From eqns.(51)-(43), the dual problem becomes

$$\min P(y) = (\bar{\mu}^1)^T y_2^1 + (\bar{\mu}^2)^T y_2^2 \tag{44}$$

subject to

$$\begin{pmatrix} I & I & 0 & 0 \\ \Delta & 0 & -\Lambda^1 & 0 \\ 0 & \Delta & 0 & -\Lambda^2 \\ -Z^1 & 0 & 0 & 0 \\ 0 & -Z^2 & 0 & 0 \\ 0 & 0 & J^1 & 0 \\ 0 & 0 & 0 & J^2 \end{pmatrix} \begin{pmatrix} y_1^1 \\ y_1^2 \\ y_2^1 \\ y_2^2 \end{pmatrix} \begin{matrix} \geq \tilde{x} \\ = 0 \\ = 0 \\ \geq 0 \\ \geq 0 \\ \geq 0 \\ \geq 0 \end{matrix} \tag{45}$$

From the first three constraints, we have $\Lambda^1 y_2^1 + \Lambda^2 y_2^2 \geq \Delta\tilde{x}$. The last two are nonnegative constraints. The other constraints are just redundant. The problem can be transformed into a simplified version

DPn:

$$\min P(y) = (\bar{\mu}^1)^T y_2^1 + (\bar{\mu}^2)^T y_2^2 \tag{46}$$

subject to

$$\Lambda^1 y_2^1 + \Lambda^2 y_2^2 = \Delta\tilde{x} \tag{47}$$

$$y_2^1, y_2^2 \geq 0 \tag{48}$$

where constraint (47) becomes an equation because for the minimization problem DPn with all the coefficients positive, the constraint is always binding. By observing DPn, we notice that y_2^1 and y_2^2 can be interpreted by the OD flows respectively belonging to origin 1 and origin 2. Thus the dual problem of LPn is actually equivalent to a **Transportation Problem**: finding the optimal pattern of the distribution of goods from several points of origin to several different destinations, with the fixed OD travel costs. In addition, if the solution of DPn is greater than the total system cost under SO, the SO and PI credit scheme is self-sustainable, and vice versa.

Theorem 2. *On a multi-origin network with fixed demands, the MRPI problem is equivalent to a single-commodity transportation problem with the same demand to each node and the OD travel costs equal to the OD travel disutilities at NTE.*

Thus it becomes very easy to find out if the pareto-improving and revenue-neutral credit scheme exists. We just need to solve DPn and compare the value to the total system cost under SO. If it's greater, the credit scheme exists, and vice versa. Since the transportation problem is a special case of a linear program, it can be readily solved by any linear programming algorithms, like the simplex method. However, the special structure of it allows us to solve it by faster, more specialized algorithms than the simplex. The only difficulty of solving the original problem LPn is to find Z^k, the origin-specific zero-flow indicators, since it is not explicitly given by the traditional traffic assignment software packages. Fortunately, there is a convenient way to calculate it. The procedure has been proposed by Dial [8], which follows a simple logic: finding the origin-stratified node potentials $\tilde{\pi}_i^k$ under the system-optimal marginal-cost arc credit rates ρ^m and corresponding arc flow pattern, $\tilde{x}$. Then

$$z_{ij}^k = \begin{cases} 0, \text{ if } \tilde{x}_{ij} + \rho_{ij}^m + \tilde{\pi}_i^k - \tilde{\pi}_j^k = 0, \\ 1, \text{otherwise.} \end{cases} \tag{49}$$

5.2. Lowest credit rates

Actually the MRPI problem may have multiple solutions. All of them can maximize the total revenue with decreasing OD travel disutilities but the credit rate levels associated with the arcs may vary quite differently. For example, if one path contains two arcs which are complementary to each other, one can raise the credit rate on one arc as high as she/he wants and just reduce the credit rate on the other one correspondingly, without changing the travel cost of the whole path. Practically, people may want to find the SO and PI credit rate set with the lowest-possible absolute values. To find the lowest credit rates we can follow two steps and in each step we need to solve a linear program: In the first step, we solve the dual problem of MRPI. We assume the value of MRPI is P_{min} and the total SO travel cost is $\tilde{TC}$ We then set

$$TC = \begin{cases} \tilde{TC}, \text{ if } P_{min} \geq \tilde{TC}, \\ P_{min}, \text{otherwise.} \end{cases} \tag{50}$$

We also define another variable $m \geq 0$ as the upper-bound of the credit rates, i.e. $\theta \leq m + \tilde{t}$; In the next step, to find the lowest credit rates we solve the following linear program

LCPI:

$$\min_{(\theta,\pi,s,\lambda,m)} m \tag{51}$$

subject to

$$\begin{pmatrix} I & \Delta^T & 0 & -Z^1 & 0 & 0 & 0 & 0 \\ I & 0 & \Delta^T & 0 & -Z^2 & 0 & 0 & 0 \\ 0 & (-\Lambda^1)^T & 0 & 0 & 0 & J^1 & 0 & 0 \\ 0 & 0 & (-\Lambda^2)^T & 0 & 0 & 0 & J^2 & 0 \\ \tilde{x}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{-1} \end{pmatrix} \begin{pmatrix} \theta \\ \pi^1 \\ \pi^2 \\ s^1 \\ s^2 \\ \lambda^1 \\ \lambda^2 \\ m \end{pmatrix} \begin{matrix} = 0 \\ = 0 \\ = \bar{\mu}^1 \\ = \bar{\mu}^2 \\ = TC \\ \leq \tilde{t} \end{matrix} \tag{52}$$

$$\theta, s, \lambda, m \geq 0 \tag{53}$$

where **-1** is a column vector with all the entries equal to -1. The value of the problem LCPI actually gives the lowest-possible upper-bound of the credit rates that are either with the minimum subsidy or revenue-neutral. And because the upper-bound of the credit rate is minimized based on the maximized revenue, implicitly, the lower-bound of the credit rate is also limited. The credit scheme itself remains socially-optimal and pareto-improving.

5.3. A numerical example

For comparison, we use the the multi-origin network from Dial [8] in our numerical example. The network node-OD incidence matrix, the system-optimal arc time and marginal-cost tolls are respectively shown in Table 1 and Figure 3.

Table 1: OD matrix

O\D	3	4	Sum
1	10	20	30
2	30	40	70
Sum	40	60	

Figure 4 shows the NTE arc flows.

By solving NTE we have

$$\begin{pmatrix} \bar{\mu}_{13} \\ \bar{\mu}_{14} \\ \bar{\mu}_{23} \\ \bar{\mu}_{24} \end{pmatrix} = \begin{pmatrix} 24.85 \\ 23.70 \\ 24.16 \\ 25.03 \end{pmatrix} \tag{54}$$

Substituting the NTE OD travel costs and OD demands into DPn, we have

$$\min P(y) = 24.85 y_{13} + 23.70 y_{14} + 24.16 y_{23} + 25.03 y_{24} \tag{55}$$

subject to

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{13} \\ y_{14} \\ y_{23} \\ y_{24} \end{pmatrix} = \begin{pmatrix} 30 \\ 70 \\ 40 \\ 60 \end{pmatrix} \tag{56}$$

$$y \geq 0 \tag{57}$$

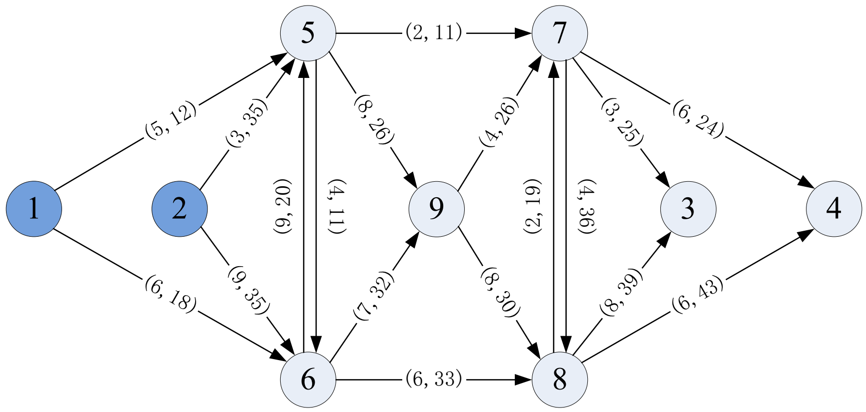

Figure 3: The 9-node network

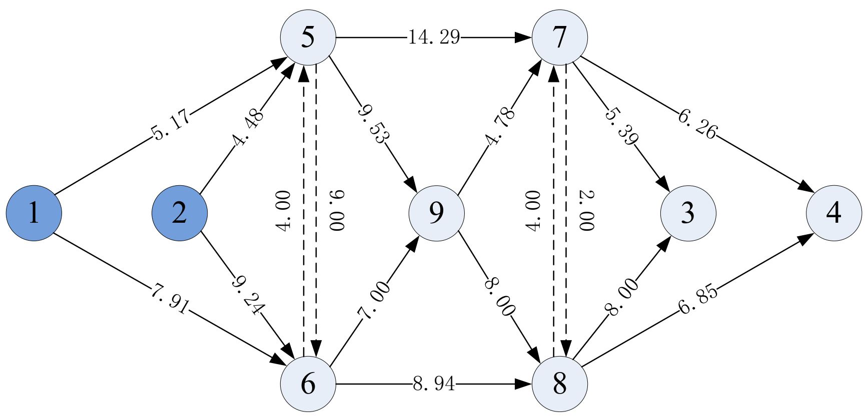

Figure 4: NTE arc time and node potentials

The value to the problem is

$$P_{min}(y) = 2428.3 \tag{58}$$

which is also the value of the primal problem LPn. By solving the SO problem we know that the total system-optimal travel cost is $2253.9 < P_{min}$, which implies that for this 9-node network, the pareto-improving and revenue-neutral credit scheme exists. And the maximum revenue the credit scheme is able to collect is $2428.3 - 2253.9 = 174.4$.

To find the credit rates, we first obtain the origin-specific flow-bearing subgraphs and node potentials, which are shown in Figure 5. Without loss of generality, we set the potentials of the origin nodes to be 0.

The solution to LPn is not unique, which is good because this leaves us more room to achieve other goals, like pursuing the lowest possible credit rates, the credit scheme with least number of arcs to be charged or the credit scheme with equity or emission constraints.

Figure 6 shows the solution with the lowest-possible credit rates, which, as we have discussed, is the solution to LCPI. All the credit rates in the network are between -4.33 and 2.39. Compared with the MC toll and MR toll in Table 2, we observe that the credit scheme produces a nonincreasing travel cost for every OD pair. The revenue of the credit scheme is

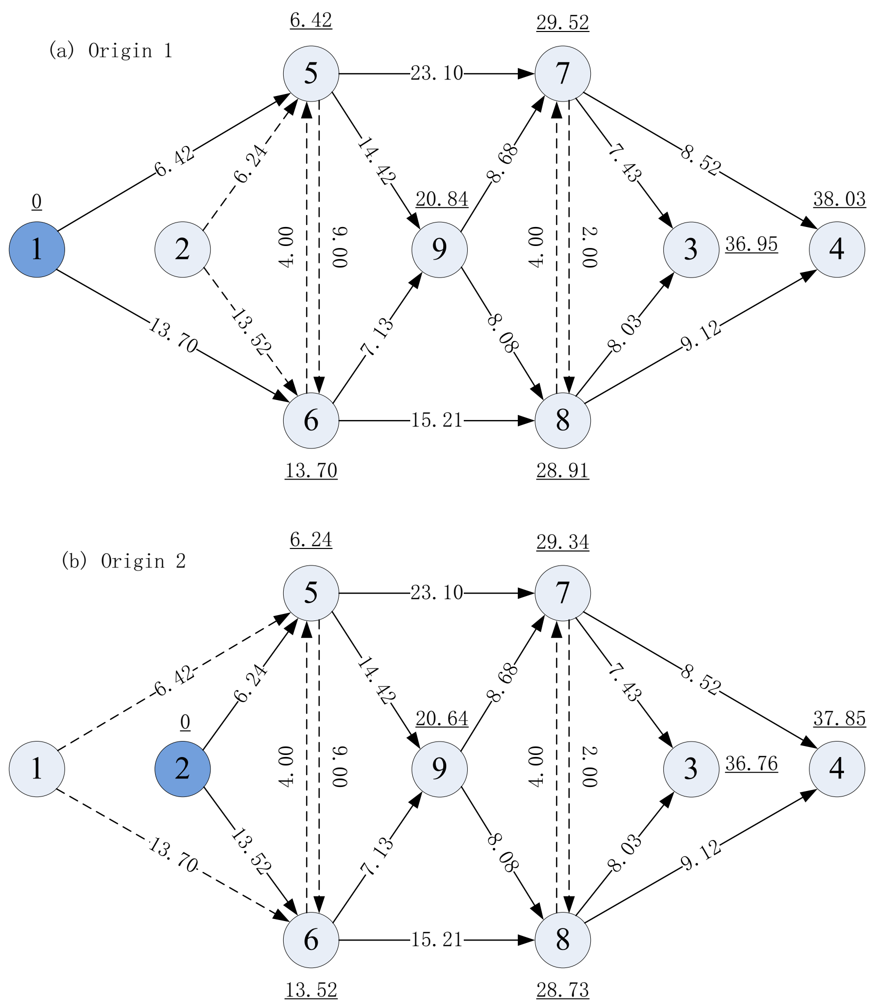

Figure 5: Subgraphs, node potentials and generalized cost under MC toll

zero.

It's worth noting that in this case if we restrict the credit rate to be nonnegative, making the problem to be a SO and PI toll problem, the primal problem becomes infeasible. Therefore, this 9-node network does provide a counter example to show that the nonnegative SO and PI toll does not always exist in a multi-origin, multi-destination network.

6. Conclusions

This paper investigate the properties of an innovative credit-based pricing scheme on a transportation network with fixed OD demands. We have two major findings from this study: first, for one-origin or one-destination network, we can always find a credit scheme to be system-optimal, pareto-improving and revenue-neutral; second, for multi-origin and multi-destination network, the SO and PI credit scheme can be calculated by solving the MRPI problem. The credit scheme may not always be self-sustainable, depending on the network structure. The dual of MRPI is equivalent to the transportation problem, so that

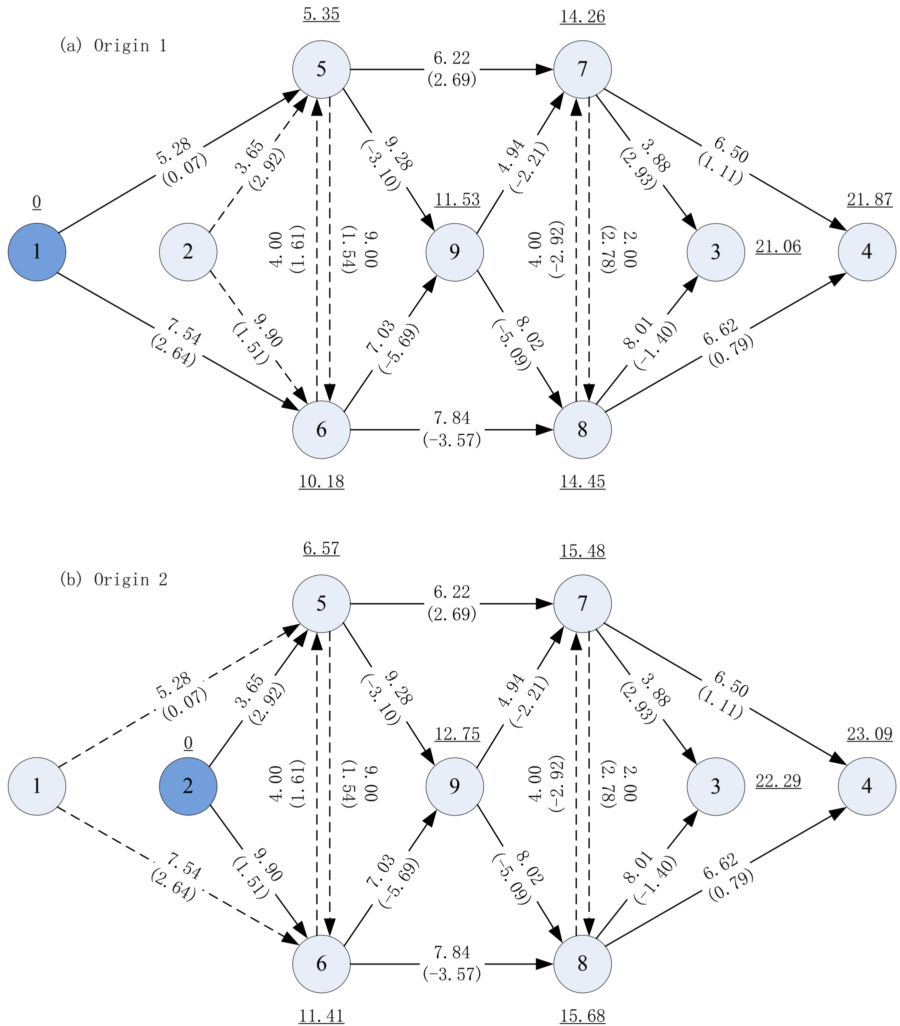

Figure 6: The pareto-improving and revenue-neutral credit rates and node potentials

we can solve the MRPI problem much faster than the traditional simplex method.

Several advantages can be found of the ABC credit scheme over traditional tolling schemes where the tolls are restricted to be non-negative. First, the additional freedom allowed by offering a credit on some arcs let us to achieve multiple planning objectives all at once. Second, the resulting travel disutility for each OD pair is usually lower in the credit scheme than in a tolling scheme, and third, the credit scheme offers a natural (anonymous) way of distributing the revenue that can be less controversial, thus improving its chances of public acceptance.

The study of the arc-based credit system is still in its infancy. Future extensions of this study may discover more potentials of this system and lead to broader applications. We are currently pursuing the following extensions:

1. *Bi-objective credit scheme design.* We know that for a mult-origin network, the pareto-improving credit scheme cannot always be self-sustainable. Under extreme situations the extra subsidy could be very high. Thus the planner may consider to reduce the subsidy

Table 2: Comparison of OD travel disutilities: NTE, MC Toll, MR Toll and Credit Scheme

OD pair	NTE	MC Toll	MR Toll	Credit Scheme	
1-3	24.85	36.95	30.6	21.08	0.85
1-4	23.70	38.03	29.21	23.70	1.00
2-3	24.16	36.76	32.96	20.92	0.87
2-4	25.03	37.85	31.57	23.54	0.94

by allowing a tolerable increase in OD travel disutility. Then the model has to be extended to tackle the following dual objectives: minimizing the subsidy and the OD travel disutility.

2. *Multi-class travelers and elastic demand.* Needless to say, considering value-of-time differentiation and elastic demand make the problem more complicated. For elastic demand, the arc flows can no longer be optimal under the credit scheme because the OD demands are changed. And for multi-class travelers the system planner has to make a balance between travel time cost and monetary cost. And it will be harder to achieve the goal of pareto-improving.

3. *Other planning purposes.* In most cases the SO, PI and revenue-neutral credit scheme is not unique, providing extra flexibility for the system planner to realize further planning purposes, like finding the maximum number and optimal location of the free arcs. Capacity and environmental factors can also be involved as additional constraints.

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